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The current source-function technique solution of electromagnetic scattering from a half plane

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Integral equations for electromagnetic scattering problems have usually been based on procedures attributed to Pocklington or Hallén. Integral equations for the unknown induced sources are obtained after utilizing the vector and scalar potentials as an intermediary to relate the fields and currents. This paper utilizes an alternative field-source relationship to obtain an integral equation which retains the integral operator with the simple kernel of Hallén's equation as well as the simple forcing function of Pocklington's equation. Further benefits of this formulation are yet to be determined. The unknown function in the integral equation is called the current source-function since it is the forcing function in an inhomogeneous differential equation for the current induced on the scatterer. The purpose of the work presented here is to develop further this new technique by applying it to a classical problem. Hence, the solution for the current induced on a perfectly conducting half-plane by a plane-wave H-polarized incident field is developed.

1. INTRODUCTION

Current source-function (CSF) *technique* is the name applied here to a formulation for electromagnetic scattering problems which does not require the use of the conventionally used vector and scalar potentials. It is based on a direct relationship between the electric field $\boldsymbol{E}$ and the current density $\boldsymbol{J}$. For harmonic time dependence $e^{-i\omega t}$,

$$i\omega\epsilon\,(\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E}) = \nabla\nabla\cdot\boldsymbol{J} + k^2\boldsymbol{J} \equiv \boldsymbol{U} \tag{1}$$

where ω is the radian frequency, k is the (free-space) wave number, and ϵ is the permittivity of the medium. Since $\boldsymbol{U} \equiv \nabla\nabla\cdot\boldsymbol{J} + k^2\boldsymbol{J}$ appears as a driving function in this inhomogeneous differential equation for the current, it is called the vector current source-function. If $\boldsymbol{U}$ is known, then $\boldsymbol{E}$ can be found from

$$i\omega\epsilon\,\boldsymbol{E} = \boldsymbol{U} * \Phi \tag{2}$$

where Φ is a solution of $(\nabla^2 + k^2)\Phi = \delta(\boldsymbol{r})$. The $*$ represents convolution and $\delta(\boldsymbol{r})$ is the Dirac delta function. This paper develops the procedures required to use the CSF method by applying the technique to the problem of scattering of an electromagnetic wave from a conducting half plane.

2. THE HALF-PLANE PROBLEM

The half-plane problem in electromagnetic theory was first solved by *Sommerfeld* [1896]. Either an E-polarized plane wave (E vector parallel to the edge as in Figure 1) or an H-polarized plane wave (H vector parallel to the edge as in Figure 2) is incident upon a perfectly conducting half plane extending from zero to infinity along the positive z axis in the $x = 0$ plane. It is desired to find either the fields surrounding the conductor or the current on it. The problem can be attacked from the differential equation viewpoint, starting with the wave equation and either a Dirichlet boundary condition (E polarization) or a Neumann boundary condition (H polarization). *Lord Rayleigh* [1897] and, much later, *Bouwkamp* [1946] observed that the solution to the wave equation with a particular boundary condition (either the Neumann or the Dirichlet) is not unique. Therefore, the half-plane problem subject to the conditions stated above alone has an infinite number of solutions. Application

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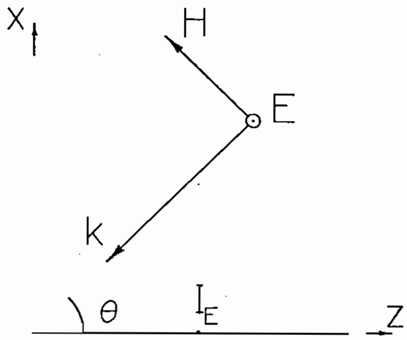

Fig. 1. *E*-polarized plane wave.

of a further condition, the edge condition, gives a unique result.

Over the last 10 years, the solution of electromagnetic scattering and radiation problems using integral equations has become routine. Most of this work has proceeded from equations of Hallén or Pocklington type, which have been studied extensively. The Pockington formulation requires that an integro-differential equation be solved. For the H-polarization half-plane problem, for example, this takes the form of a second-order differential operator applied to the convolution integral of the unknown current and a Hankel function kernel $H_0^{(1)}(k|z|)$ where $H_0^{(1)}(t) = J_0(t) + iY_0(t)$. The right-hand side is proportional to the tangential incident field evaluated at $x = 0, E_z^i = \sin\theta\, e^{-ikz\cos\theta}$ (see Figure 2). The integro-differential equation is

$$\left(\frac{d^2}{dz^2} + k^2\right) \int_0^\infty I_H(z')\, H_0^{(1)}(k|z - z'|)dz'$$
$$= (4k/Z_0) \sin\theta\, e^{-ikz\cos\theta}, \quad z > 0 \qquad (3)$$

Here, $k = \omega/c$ where ω is the radian frequency, c is the speed of light, and Z_0 is the intrinsic impedance of the (free-space) medium. A second

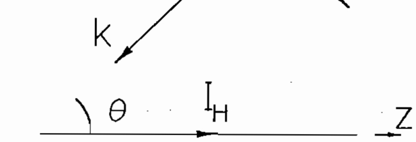

Fig 2. *H*-polarized plane wave.

formulation, the Hallén formulation, involves removing the differential operator in (3) by Green's function techniques. This usually results in a right-hand side involving an integral of the incident field. A third possibility, which has not received a great deal of attention, was previously suggested by one of the authors [*Mayes*, 1972]. This integral equation formulation utilizes (1) and (2) to relate the E field and the unknown current density I_H. The details of this derivation are presented in appendix A. The integral equation can also be found from (3) by taking the differential operator under the integral sign and placing it on the unknown current density $I_H(z')$. The integral equation formulation is

$$\int_0^\infty u_H(z')\, H_0^{(1)}(k|z - z'|)dz'$$
$$= (4k/Z_0) \sin\theta\, e^{-ikz\cos\theta}, \quad z > 0 \qquad (4)$$

where

$$u_H(z') = d^2 I_H/dz'^2 + k^2 I_H \qquad (5)$$

Note that (4) is an integral equation with the same kernel as (3), but without the differential operator on the outside of the integral. The right-hand side of (4) is proportional to the incident field rather than to an integral of the incident field as is the case in the Hallén formulation. The unknown quantity in (4), however, is not the induced current, but rather an auxiliary function, called the current source-function, from which the current can be obtained by solving the inhomogeneous differential equation of (5). Related work has been done by W. A. Johnson and D. G. Dudley (personal communication, 1977) and *Walsh* [1976].

It might appear that the solution to (4) is unique, but why would the wave equation approach give an infinite number of solutions, and the integral equation approach only one? The answer has to do with the tacit assumption that the solution (the unknown u_H) to an integral equation is integrable in some sense. Differentiating the current I_H according to (5) may mean that $u_H(z)$ is not integrable in the usual sense. This would also cast doubt on the validity of taking the derivative operator under the integral sign.

3. DIVERGENT INTEGRALS AND THE FINITE PART

Bouwkamp [1954, pp. 40, 68–69] points out that if the d^2/dz^2 operator of (3), for example, is taken under the integral sign and placed on the kernel, then the resulting kernel is nonintegrable. Such an integral can be assigned a meaning, however, by introducing the concept of the finite part. This concept dates back to *Cauchy* [1826] who used it to assign a meaning to the gamma function for negative values of the argument. *Hadamard* [1923] extends the concept to the multidimensional case. A lengthy bibliography and a general discussion of the history of the finite part of divergent integrals is given by *Bureau* [1955, pp. 143–146]. Both Hadamard and Bureau use the finite-part concept in connection with solving partial differential equations.

Hadamard [1923, pp. 134–141] introduces the theory of the finite part of divergent integrals and discusses several examples at length. Two examples of finite-part integration which are relevant to the present work are given here. As a first example, consider the integral

$$\int_a^b \frac{dt}{(b-t)^{3/2}} = \frac{2}{(b-t)^{1/2}}\Bigg|_a^b \tag{6}$$

The right-hand side goes to infinity when it is evaluated at the upper limit. The finite part of this expression is found by retaining only the value obtained when the right-hand side is evaluated at the lower limit. Hence,

$$Fp\int_a^b \frac{dt}{(b-t)^{3/2}} = \frac{-2}{(b-a)^{1/2}} \tag{7}$$

where the letters *Fp* indicate the finite part.

By following a similar procedure, using the assumption that $I_H \sim z^{1/2-\tau}$, $0 \le \tau < 1/2$, as $z \to 0$, and the identity

$$(\partial/\partial z)\, H_0^{(1)}(|z-t|) = -(\partial/\partial t) H_0^{(1)}(|z-t|) \tag{8}$$

and integrating by parts, it can be shown that the derivative operator of (3) may be taken under the integral sign as long as the finite part is taken of the resulting integral. That is,

$$\frac{d^2}{dz^2}\int_0^\infty I_H(t)\, H_0^{(1)}(k|z-t|)dt = Fp\int_0^\infty \left(\frac{d^2}{dt^2} I_H(t)\right) H_0^{(1)}(k|z-t|)dt, \quad z > 0 \tag{9}$$

where

$$Fp\int_0^\infty \left(\frac{d^2}{dt^2} I_H(t)\right) H_0^{(1)}(k|z-t|)dt = \lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty \left(\frac{d^2}{dt^2} I_H\right) H_0^{(1)}(k|z-t|)dt + [(d/dt)\, I_H(t)]\Big|_\epsilon H_0^{(1)}(k|z|)\right\} \tag{10}$$

The last term must be included because dI_H/dt is assumed to have $t^{-1/2-\tau}$ edge behavior with $0 \le \tau < 1/2$.

In this way, the integral equation (3) may be written in the form

$$Fp\int_0^\infty u_H(z')\, H_0^{(1)}(k|z-z'|)dz' = (4k/Z_0)\sin\theta\, e^{-ikz\cos\theta}, \quad z > 0 \tag{11}$$

where u_H is given by (5). This integral equation may be shown to have an infinite number of solutions, which Lord Rayleigh and Bouwkamp also found to be the case in the wave equation approach to the problem. Thus, the use of the finite part integral equation correlates better with the wave equation approach than does the use of the ordinary integral equation. A derivation of (11) starting with (1) and (2) is given in appendix A. A wave equation approach to the current source-function technique solution of the half-plane problem is given by *Hanson* [1976].

The result of (9) is more easily proven when the convolution equations of Schwartz distribution theory are used instead, because then it is easily shown that

$$(d^2/dz^2)(I * K) = (d^2 I/dz^2) * K = I * (d^2 K/dz^2) \tag{12}$$

where I and K are appropriate distributions and the $*$ represents convolution. Schwartz distribution theory is based on finite part integration for some definitions. The Schwartz distribution theory interpretation of the problem is discussed in detail by *Hanson* [1976].

4. SOLUTION OF THE INTEGRAL EQUATION FOR $u_H(z)$

In this section, the finite-part integral equation (11) is solved for u_H. Methods for solving finite-part integral equations have been studied by several authors. *Butzer* [1959] and *Boehme* [1963] use operational calculus to study the finite part of divergent convolution integrals. They both treat finite-part singular integral equations of Volterra type. *Wiener* [1962] treats linear finite-part integral equations of Fredholm second kind and Volterra types. In a series of over thirty papers published over the last fifteen years (generally published in *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg, Mathematische Nachr.*, or *Beitr. Anal.*), he and his colleagues treat many finite-part integral equations. However, it appears that Fredholm finite-part integral equations of the first kind with Hankel function kernels, such as (11), have not yet been treated explicitly. Work on a similar integral equation except with a $K_0(t)$ kernel has been reported by *Belward* [1972], however. For the finite-part equation (11), one or more solutions of the homogeneous equation

$$Fp\int_0^\infty w(z')\,H_0^{(1)}(k|z-z'|)\mathrm{d}z' = 0, \quad z>0 \tag{13}$$

may be added to the solution of the ordinary integral equation (equation (11), but without the *Fp* prefix) to obtain another solution. The number of solutions that may be added is related to the edge condition on the current source-function. This simply states that the maximum allowed edge singularity is $z^{-3/2}$, that is,

$$u_H(z) = O(z^{-3/2}), \quad z \to 0^+ \tag{14}$$

This is obtained from the edge condition on the H-polarization current, which is [*Mittra and Lee*, 1971, pp. 4-11]:

$$I_H(z) = O(z^{1/2}), \quad z \to 0 \tag{15}$$

The homogeneous integral equation (13) can be solved by several methods. One technique follows and another, the Wiener-Hopf technique, is detailed by *Hanson* [1976].

Both the integral equation for the E-polarization half-plane current $I_E(z)$ and the integral equation for the ordinary part of the H-polarization current source-function $u_H(z)$ may be written as

$$\int_0^\infty \mathrm{v}(z')\,H_0^{(1)}(k|z-z'|)\mathrm{d}z' = \frac{4k}{Z_0}\,\mathrm{e}^{-ikz\cos\theta}, \quad z>0. \tag{16}$$

Note that the right-hand side is the same as in (11) except for a $\sin\theta$ factor. The exact solution of this equation is [*Noble*, 1958, p. 228]:

$$\mathrm{v}(z) = 1_+(z)\,\frac{4\,k^2\,\mathrm{e}^{-i\pi/4}}{Z_0\sqrt{\pi}}\left\{\frac{i\sin(\theta/2)}{\sqrt{2}}\,\frac{\mathrm{e}^{ikz}}{\sqrt{kz}} + \sin\theta\,\mathrm{e}^{-ikz\cos\theta}\int_0^{[kz(1+\cos\theta)]^{1/2}} \mathrm{e}^{it^2}\mathrm{d}t\right\} \tag{17}$$

where $1_+(z)$ is the unit step function. This is seen to have $z^{-1/2}$ edge behavior. Since the edge condition for the H-polarization current source-function is $u_H(z) = O(z^{-3/2})$ as $z \to 0$, the function $\sin\theta\,\mathrm{v}(z)$ alone does not have to be the total solution of (11). Solutions of (13) which at most behave as $z^{-3/2}$ as $z \to 0$ may be added to $\sin\theta\,\mathrm{v}(z)$ without violating any of the conditions of the problem.

To find a solution of (13), consider the integral equation

$$\int_0^\infty f(z')\,H_0^{(1)}(k|z-z'|)\mathrm{d}z' = \frac{4k}{Z_0}, \quad z>0 \tag{18}$$

The solution $f(z)$ of this integral equation is given by $\mathrm{v}(z)$ of (17) with $\theta = 90°$. Applying the operator $L = \mathrm{d}/\mathrm{d}z$ to both sides of (18) yields

$$Fp\int_0^\infty Lf(z')\,H_0^{(1)}(k|z-z'|)\mathrm{d}z' = 0, \quad z>0 \tag{19}$$

so that a solution of the homogeneous finite part integral equation of (13) is

$$w(z) = \frac{\mathrm{d}\,f(z)}{\mathrm{d}z} = -\frac{k^3\,\mathrm{e}^{i\pi/4}}{Z_0\sqrt{\pi}}\,\frac{\mathrm{e}^{ikz}}{(kz)^{3/2}}\,1_+(z) \tag{20}$$

Clearly, any derivative of this is also a solution of (13). Equation (20) is the only solution which satisfies the edge condition on $u_H(z')$.

5. THE CONSISTENCY CONDITION

The complete or total solution of the finite part integral equation of (11) for $u_H(z)$ becomes

$$u_H(z) = \sin\theta\,\mathrm{v}(z) + A\,w(z) \tag{21}$$

where A is (at this point) an arbitrary constant and $\mathrm{v}(z)$ is given by (17).

In appendix B it is shown that the system

$$\mathrm{d}^2 I_H/\mathrm{d}z^2 + k^2 I_H = u_H(z), \quad 0<z<\infty \tag{22}$$

with $I_H(0) = 0$ and I_H satisfying an appropriate condition as $z \to +\infty$, has no solution unless *the consistency condition*

$$Fp \int_0^\infty u_H(z)\, e^{ikz} dz = 0 \tag{23}$$

is satisfied. Substituting (21) in (23), one obtains

$$0 = \int_0^\infty u_H(z)\, e^{ikz} dz$$

$$= \sin\theta \int_0^\infty v(z)\, e^{ikz} dz + A \int_0^\infty w(z)\, e^{ikz} dz \tag{24}$$

where the integrals are to be finite part integrals when necessary. The solution for A becomes

$$A = -\left[\sin\theta \int_0^\infty v(z)\, e^{ikz} dz\right] \Big/ \int_0^\infty w(z)\, e^{ikz} dz \tag{25}$$

Therefore, there is only one value of A for which a solution for I_H exists. The integrals of $w(z)$ and $v(z)$ with respect to e^{ikz} become

$$\int_0^\infty v(z)\, e^{ikz} dz = \frac{2ik}{Z_0 \sin(\theta/2)}, \quad \theta \neq 2n\pi \tag{26}$$

and

$$Fp \int_0^\infty w(z)\, e^{ikz} dz = \frac{2\sqrt{2}\, k^2}{Z_0} \tag{27}$$

Substituting these in (25) and simplifying yields

$$A = [\sqrt{2} \cos(\theta/2)]/ik \tag{28}$$

The unique solution for $u_H(z)$ becomes

$$u_H(z) = \sin\theta\, v(z) + \frac{\sqrt{2} \cos(\theta/2)}{ik} w(z) \tag{29}$$

This is the required complete solution for $u_H(z)$. The current I_H is found in the next section.

6. CURRENT IN TERMS OF THE CURRENT SOURCE-FUNCTION

The expression for u_H, the H-polarization current source-function, is given by (29). The relation between u_H and I_H is given by (22). I_H may be found in terms of u_H by using Green's function techniques. This result is

$$I_H(z) = \int_0^\infty u_H(z')\, g(z,z') dz' \tag{30}$$

where finite-part integration is assumed (if necessary) and $g(z,z')$ is a Green's function for the operator $\mathscr{L} = (d^2/dz^2 + k^2)$. This Green's function may be found by solving the distributional inhomogeneous second-order differential equation

$$\mathscr{L}^* g = \delta(z - z'), \quad 0 < z,z' < \infty,$$

$$g \text{ outgoing as } z' \to \infty \tag{31}$$

subject to the adjoint boundary conditions where $\mathscr{L}^*(= \mathscr{L})$ is the adjoint of $\mathscr{L}$. Only a radiation type adjoint boundary condition is required at infinity. It is shown in appendix B that no particular adjoint boundary condition is required at $z' = 0$. If the condition $g(z,z' = 0) = 0$ is imposed on g, then one obtains

$$g(z,z') = (1/2ik)(e^{ik|z-z'|} - e^{ik(z+z')}) \tag{32}$$

The current I_H is found from (30). It is interesting to note that since

$$\int_0^\infty u_H\, e^{ikz'} dz' = 0$$

by the consistency condition of (23), then the current may also be written as

$$I_H(z) = \int_0^\infty u_H(z')\, E(z-z') dz' \tag{33}$$

where $E(z) = (1/2ik)e^{ik|z|}$. This is a "fundamental solution" for the operator $\mathscr{L}$ as is used in the theory of distributions. The current I_H is the convolution of u_H with E. Thus, the integral in (30) has been reduced to the convolution integral in (33).

Substituting (29) for u_H in (33), one obtains

$$I_H(z) = \frac{\sin\theta}{2ik} \int_0^\infty v(z')\, e^{ik|z-z'|} dz'$$

$$+ \frac{\sqrt{2} \cos(\theta/2)}{ik(2ik)} Fp \int_0^\infty w(z')\, e^{ik|z-z'|} dz' \tag{34}$$

The integrals in this expression are found to be

$$\int_0^\infty v(z')\, e^{ik|z-z'|} dz' = \frac{4k e^{i\pi/4}}{Z_0 \sqrt{\pi}}$$

$$\cdot \left(\frac{e^{-ikz}[F_2(\infty) - 1_+(z) F_2(2kz)]}{\sin(\theta/2)} + \frac{2}{\sin\theta} \cdot 1_+(z) e^{-ikz\cos\theta} F_2[kz(1+\cos\theta)] \right) \tag{35}$$

and

$$Fp \int_0^\infty w(z') e^{ik|z-z'|} dz' = \frac{4k^2 e^{-i\pi/4}}{Z_0 \sqrt{\pi}} \cdot \sqrt{2}\, e^{-ikz} [F_2(\infty) - 1_+(z) F_2(2kz)] \tag{36}$$

where

$$F_2(x) = \int_0^{\sqrt{x}} e^{it^2} dt \tag{37}$$

and $F_2(\infty) = \sqrt{\pi}\, e^{i\pi/4}/2$. Substituting (35) and (36) into (34) and simplifying gives

$$I_H(z) = [4e^{-i\pi/4} /(Z_0 \sqrt{\pi})] 1_+(z) \cdot e^{-ikz\cos\theta} \int_0^{[kz(1+\cos\theta)]^{1/2}} e^{it^2} dt, \quad (\mathrm{A/m})/(\mathrm{V/m}) \tag{38}$$

From (33), it is important to note that

$$I_H(z) = \frac{e^{-ikz}}{2ik} \int_0^\infty u_H(z') e^{ikz'} dz' = 0, \quad z < 0 \tag{39}$$

The integral is identically zero for all negative z. This is true because of the consistency condition of (23). Thus,

$$I_H(z) = \int_0^\infty u_H(z') E(z - z') dz', \quad -\infty < z < \infty \tag{40}$$

The current is given by the integral for all values of z.

7. CONCLUSIONS

The main purpose of this paper has been to develop the procedures for applying the current source-function technique to electromagnetic scattering from a perfectly conducting half plane. The solution to the integral equation (11) is found by adding the solution of the ordinary integral equation to a solution of the homogeneous finite-part integral equation such that the edge condition is not violated.

A constant multiplying the homogeneous solution is found by enforcing the consistency condition (23) which is developed in appendix B. The current is found by inverting the $d^2/dz^2 + k^2$ operator to obtain

$$I_H = u_H * E, \quad \text{all } z$$

The fundamental solution E may be used instead of a Green's function because of the consistency condition. The result for I_H, given by (38), is identical with the result obtained by other methods.

Not only has the exact solution been obtained using the CSF technique as shown here, but also the same procedure applied numerically has yielded data closely in agreement with the above results [*Hanson*, 1976].

The success of the CSF technique application to the half-plane problem is a positive indication of its feasibility for arbitrary perfectly electrically conducting (p.e.c.) objects (even those with sharp edges). The values of any divergent integrals which are introduced by sharp edges could apparently be defined by using Hadamard's finite part. The CSF technique appears to be more straightforward for p.e.c. objects with suitably well-behaved current distributions. The general nature of the CSF technique is indicated by *Hanson* [1976, pp. 162–165], wherein a brief outline of the current source-function technique for three-dimensional time domain problems is given. Thus, there are many possibilities for application of the CSF technique which remain to be explored.

APPENDIX A: DERIVATION OF (11)

The integral equation (11) can also be derived directly from Maxwell's equations. It is the purpose of this appendix to show how this can be done for the harmonic problem with $e^{-i\omega t}$ time dependence where ω is the radian frequency. Assuming that any conducting inhomogeneities in a region have been replaced with the induced electric current density $\boldsymbol{J}$ acting in a homogeneous medium with constitutive parameters ϵ and μ, Maxwell's equations for the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ produced by these currents may be used to show that

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = -\frac{1}{y} \{\nabla\nabla \cdot \boldsymbol{J} + k^2 \boldsymbol{J}\} \equiv -(1/y)\boldsymbol{U} \tag{41}$$

where $\boldsymbol{U}$ is the vector current source-function,

$y = -i\omega\epsilon$, and $k^2 = \omega^2\mu\epsilon$. *Collin* [1960, p. 21, eq. (51b)] gives this relation between $\boldsymbol{E}$ and $\boldsymbol{J}$. Therefore, this direct relationship is not new, but its application to integral equations apparently is. Solving for the electric field $\boldsymbol{E}$ due to the current density $\boldsymbol{J}$ yields

$$\boldsymbol{E} = \frac{1}{4\pi y}\iiint_V \boldsymbol{U}(r')\frac{e^{ikR}}{R}\mathrm{d}V' \tag{42}$$

where $R = |r - r'|$ is the source-point to observation-point distance.

The following discussion is limited to the half-plane problem. A more general discussion can be found in *Hanson* [1976]. For the H polarization, the current flows in the z direction only. The current $\boldsymbol{J}_H$ can be expressed as

$$\boldsymbol{J}_H = I_H(z)\,1_+(z)\,\delta(x)\,\hat{z} \tag{43}$$

where $1_+(z)$ is the unit step function and δ is the Dirac delta function. $I_H(z)$ is the unknown current distribution. The vector current source-function for this case becomes

$$\boldsymbol{U}_H = \nabla\nabla\cdot\boldsymbol{J}_H + k^2\boldsymbol{J}_H = [\partial(I_H 1_+)/\partial z][\partial\delta(x)/\partial x]\,\hat{x} + \delta(x)(\mathrm{d}^2/\mathrm{d}z^2 + k^2)(I_H 1_+)\,\hat{z} \tag{44}$$

For a proper interpretation, these derivatives must be performed using Schwartz distribution theory. Substituting (44) in (42) yields the expression for the H-polarization scattered electric field. After using the fact that

$$\int_{-\infty}^{\infty}\frac{e^{ik(x^2+\eta^2+z^2)^{1/2}}}{(x^2+\eta^2+z^2)^{1/2}}\mathrm{d}\eta = i\pi H_0^{(1)}[k(x^2+z^2)^{1/2}] \tag{45}$$

where $H_0^{(1)}$ is the Hankel function, the expression becomes

$$\boldsymbol{E}^s = -\hat{x}\frac{Z_0}{4k}\int_{z'}\int_{x'}\frac{\partial(I_H 1_+)}{\partial z'}\frac{\partial\delta(x')}{\partial x'}\cdot H_0^{(1)}\{k[(x-x')^2+(z-z')^2]^{1/2}\}\mathrm{d}x'\,\mathrm{d}z' - \hat{z}\frac{Z_0}{4k}\int_0^{\infty}u_H(z')H_0^{(1)}\{k[x^2+(z-z')^2]^{1/2}\}\mathrm{d}z'$$

where

$$u_H(z') = \mathrm{d}^2 I_H/\mathrm{d}z'^2 + k^2 I_H \tag{47}$$

The total field $\boldsymbol{E}^t$ is equal to the sum of the incident and scattered fields or

$$\boldsymbol{E}^t = \boldsymbol{E}^i + \boldsymbol{E}^s \tag{48}$$

The integral equation for $u_H(z')$ is obtained by applying the boundary condition that the tangential electric field must be zero on the half plane. The integral equation becomes

$$Fp\int_0^{\infty}u_H(z')H_0^{(1)}(k|z-z'|)\mathrm{d}z' = \frac{4k}{Z_0}E_z^i(x=0,z) = (4k/Z_0)\sin\theta\, e^{-ikz\cos\theta}, \quad z>0 \tag{49}$$

where θ is the angle of incidence defined in Figure 2 and E_z^i is the z component of the incident field. This is just (11) of the text.

APPENDIX B: DERIVATION OF THE CONSISTENCY CONDITION

The consistency condition is required to determine the unknown constant A in (21). It is obtained by examining the conditions under which the inhomogeneous differential equation (5) relating the current source-function u_H and the current I_H has a solution. The inverse of the operator $\mathscr{L} = (\mathrm{d}^2/\mathrm{d}z^2 + k^2)$ in (5) can be represented as an integral operator with a Green's function kernel. This Green's function $g(z,z')$ is a solution of the distributional inhomogeneous differential equation

$$\mathscr{L}^*g = \delta(z-z') \tag{50}$$

subject to certain boundary conditions where $\mathscr{L}^*$ $(=\mathscr{L})$ is the adjoint of $\mathscr{L}$.

These adjoint boundary conditions are found by determining boundary conditions such that

$$\langle\mathscr{L}\eta_+,\psi\rangle = \langle\eta_+,\mathscr{L}^*\psi\rangle \tag{51}$$

where $\langle\alpha,\beta\rangle$ is shorthand for $\int_{-\infty}^{\infty}\alpha\beta\,\mathrm{d}z$. Here, η takes the place of I_H (that is, $\eta = I_H$), and ψ is the function for which the adjoint boundary conditions are to be found. The + subscript denotes that η is zero for $z < 0$. Since the current $I_H(0) = 0$ and $I_H = O(z^{1/2})$ as $z \to 0$, then $\eta(0) = 0$ and $\eta = O(z^{1/2})$ as $z \to 0$. Clearly, $\mathscr{L}\eta_+$ is allowed to be a pseudofunction (a function which requires that finite-part integration be performed on it) because

$$\mathscr{L}\eta = O(z^{-3/2}) \quad \text{as} \quad z \to 0^+ \tag{52}$$

For arbitrary ψ, this means that $\langle \mathscr{L}\eta_+, \psi \rangle$ must, in general, be interpreted as the finite-part integral

$$Fp \int_0^\infty \left(\frac{d^2\eta}{dz^2} + k^2\eta \right) \psi(z) dz \tag{53}$$

Setting the lower limit to ϵ and integrating by parts yields

$$\int_\epsilon^\infty \left(\frac{d^2\eta}{dz^2} + k^2\eta \right) \psi(z) dz = \frac{d\eta}{dz} \psi(z) \Bigg|_\epsilon^\infty - \eta(z) \frac{d\psi}{dz} \Bigg|_\epsilon^\infty + \int_\epsilon^\infty \eta(z) \left(\frac{d^2\psi}{dz^2} + k^2\psi \right) dz \tag{54}$$

Because of the allowed edge behavior of $\eta(=I_H)$,

$$d\eta/dz = O(z^{-1/2}) \quad \text{as } z \to 0^+ \tag{55}$$

Now, by assuming that $d\eta/dz$ actually has edge behavior $z^{-1/2+\tau}$, $0 \le \tau < 1/2$, as $z \to 0$, the finite-part integral can be written

$$Fp \int_0^\infty (\mathscr{L}\eta)\psi(z) dz = \lim_{\epsilon \to 0} \left\{ \int_\epsilon^\infty (\mathscr{L}\eta)\psi(z) dz + \frac{d\eta}{dz}\psi(z) \Bigg|_\epsilon \right\} = (d\eta/dz)\psi(z) \Bigg|_\infty - n(z)(d\psi/dz) \Bigg|_0^\infty + \int_0^\infty \eta(z) \mathscr{L}^* \psi dz \tag{56}$$

In order for (51) to hold, the boundary terms above must vanish. The boundary conditions to be satisfied are

$$\eta(z) d\psi/dz \to 0 \quad \text{as} \quad z \to 0 \tag{57}$$

$$(d\eta/dz)\psi(z) \to 0 \quad \text{as} \quad z \to \infty \tag{58}$$

$$\eta(z) d\psi/dz \to 0 \quad \text{as } z \to \infty \tag{59}$$

The first condition is always satisfied because $\eta(0) = 0$. The asympotic behavior of the current is

$$\eta = I_H \sim e^{-ikz\cos\theta} \quad \text{as} \quad z \to \infty \tag{60}$$

Since a plane wave incident field requires that a source be located at infinity, it is not possible to specify a radiation-type condition at plus infinity when the source of the plane wave is located there. If, however, θ (see Figure 2) is limited to be in the range $90° < \theta < 270°$, the source is located at negative infinity and an outgoing plane wave is expected as $z \to +\infty$. Once the problem has been solved under this θ limitation, θ can be extended to all angles of incidence. This initial limitation on θ allows (58) and (59) to be satisfied for a slightly lossy medium. Thus, a small imaginary part may be introduced in k such that $k = k_1 + ik_2$. This gives

$$\eta, \quad \frac{d\eta}{dz} \sim e^{k_2 z \cos\theta} \quad \text{as} \quad z \to \infty \text{ with } \cos\theta < 0 \tag{61}$$

From (57)–(59), ψ and $d\psi/dz$ must exhibit the exponential form

$$\psi, \quad d\psi/dz \sim e^{-\zeta z} \quad \text{as} \quad z \to \infty \tag{62}$$

with $\zeta > k_2\cos\theta$ in order for the boundary terms to vanish. It turns out that $\zeta = k_2$ so that this condition is always satisfied. This is the only adjoint boundary condition that is required. A condition at zero is not required for the adjoint problem since finite-part integration removes zero from consideration. Once the solution for a lossy medium has been found, that for a lossless medium can be obtained by setting $k_2 = 0$.

The solvability of the second-order differential equation $(d^2/dz^2 + k^2)y = f(z)$, $a < z < b$, such that certain boundary conditions are satisfied is closely related to the existence of solutions to the homogeneous system and to the adjoint homogeneous system. In the case of a scatterer of finite extent, solutions of the homogeneous system arise only at resonance, i.e., when the physical extent of the scatterer matches a multiple of a half wavelength of the incident field. For the semi-infinite case under consideration here, the homogeneous system has no nontrivial solutions. For nonsingular differential equations, it may be shown that if the homogeneous system has only the trivial (zero) solution, then the adjoint homogeneous system has only the trivial (zero) solution. For the details, the reader is referred to *Stakgold* [1967, Vol. I, pp. 84–85]. For problems with functions requiring finite-part integration, this is (in general) no longer true. The adjoint homogeneous system usually has solutions even though the homogeneous system has only the trivial solution.

For the half-plane problem, the homogeneous, inhomogeneous, and adjoint homogeneous systems are

homogeneous system:

$$\mathscr{L}\rho = 0, \quad 0 < z < \infty, \quad \rho(0) = 0;$$
$$\rho \to 0 \text{ exponentially as } z \to \infty \tag{63}$$

inhomogeneous system:

$$\mathscr{L}\eta = f, \quad 0 < z < \infty, \quad \eta(0) = 0;$$
$$\eta \to 0 \text{ exponentially as } z \to \infty \tag{64}$$

adjoint homogeneous system:

$$\mathscr{L}^*\psi = 0, \quad 0 < z < \infty;$$
$$\psi \to 0 \text{ exponentially as } z \to \infty \tag{65}$$

where $\mathscr{L}^* = \mathscr{L} = (d^2/dz^2 + k^2)$. As was shown previously, the adjoint homogeneous system does not have a boundary condition to be satisfied at $z = 0$. The differential equation of (63) has solutions $\{e^{-ikz}, e^{+ikz}\}$ or $\{\sin(kz), \cos(kz)\}$, but none of these satisfies both of the boundary conditions so (63) has only the trivial solution $\rho \equiv 0$. The adjoint homogeneous differential equation also has the above solutions, but in this case one of them does satisfy the given boundary condition (again assuming that $k = k_1 + ik_2$, $0 < k_2 \ll 1$, and that θ is restricted). The nonzero solution of the adjoint homogeneous system (65) is

$$\psi = e^{+ikz} \tag{66}$$

The following theorem is similar to one given by *Stakgold* [1967, Vol. I, p. 85].

THEOREM: System (64) has no solution unless the consistency condition

$$\int_0^\infty f(z)\psi(z)dz = 0$$

is satisfied for every $\psi(z)$ which is a solution of (65).

PROOF: $\mathscr{L}$ is a second-order differential operator and hence can have no more than two nonzero linearly independent homogeneous solutions. Only one of these is an outgoing wave and goes to zero as $z \to \infty$. These are the only conditions required on ψ by (65). Multiplying (64) by ψ and (65) by η, subtracting, and integrating from 0 to ∞ gives

$$\int_0^\infty (\psi\mathscr{L}\eta - \eta\mathscr{L}^*\psi)dz = \int_0^\infty f(z)\psi(z)dz \tag{67}$$

The left-hand side is zero by applying the results of (57)–(59) to (56). Therefore,

$$\int_0^\infty f(z)\psi(z)dz = 0 \tag{68}$$

must hold for every ψ that satisfies (65). Note that for the trivial solution $\psi(z) \equiv 0$, this consistency condition is always satisfied. Since $\psi = e^{ikz}$ is the only such ψ, (5) has no solution unless the consistency condition

$$Fp \int_0^\infty u_H(z)\, e^{ikz} dz = 0 \tag{69}$$

is satisfied.

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